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A PALM PROBABILISTIC TECHNIQUE ON THE ESTIMATION OF THE EXPECTED TIME TO FAILURE AT A SPECIFIED SHOCK BASED ON THE DATA OF EARLIER SHOCKS

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SUMMARY

A methodology to estimate the failure time of a component which successively gets damaged on arrival of shocks with compound Poisson inputs (weighted by Gamma distribution) based on the available information of the data relating to the earlier shocks has been evolved in this paper. The inter-arrival distribution of shocks because of compound Poisson intensities being infinitely divisible with dependent increments, use of Palm probability technique has been employed to predict the waiting time for failure. The exercise has been done both with time independent as well as time dependent Poisson inputs while assuming the distribution of the number of shocks causing the failure of the component to be geometric.

Keywords : Successive damage model; Shocks; Palm probability; Compound Poisson Inputs; Infinitely Divisible Distributions with dependent increments, Cardio Vascular Shocks.

Introduction

The importance of studies relating to successive damage model is being interestingly realised in reliability theory as well as in the survival theory of biosystems. A shock causes damage to a component of the system and a series of such shocks produces successive damages to the system ultimately leading to its complete failure. Since shocks occur in a variety of ways the human biosystem, while describing the effect of the same qualitatively as well as quantitatively appropriate probabilistic techniques are necessary to formulate the modelling pertaining to the same providing useful practical solutions to several problems of prediction and estimation. For example, consider problem of predicting the longevity of a patient who undergoes cardiovascular shocks which cause failure of the Human system on arrival of a threshold number of shocks (which is a random variable varying from one patient to another). The interarrival distribution of shocks may be assumed to occur with time independent or time dependent intensities λ or $\lambda e^{-\delta t}$. However, as pertinent in human system λ may vary from individual to individual following a certain probability distribution say, a Gamma distribution. But as a result of weighting of λ by Gamma distribution, the waiting time distribution between consecutive renewal intervals (say between i to $(i+1)$th shocks and $(i+1)$ to $(i+2)$th shocks) become correlated (Biswas and Pachal [1]; Biswas and Nair [2]. This completely distroys the renewal structure of the process.

The problem considered in this paper is therefore to evolve a methodology of predicting the expected failure time of a system which gets damaged successively on arrival of repeated shocks and ultimately fail on receipt of a threshold number of shocks; on the assumption that the renewal structure of the shocks conforms to infinitely divisible distributions with dependent increments. More precisely, given that $(r+1)$ consecutive shocks occur at times $t_0 < t_1 < t_2 < \ldots < t_r$ which are known, the problem comprises of predicting the expected time of failure of the system; say at the $(n+1)$th shock (n is a random variable), using the data of occurrence of the first $(r+1)$ consecutive shocks, when the renewal process of the interoccurrence of the shocks conforms to a dependent process.

Under the circumstanees given that a patient has experienced $(r+1)$ number of non-fatal shocks $(r = 0, 1, 2, \ldots, n-1)$, it often becomes an important problem to predict the residual survival period. The present paper is an exercise towards evolving a methodology for the same, using the technique of Palm probability (Khintchine [5]) in both the cases when the intensity is time independent as well as time dependent (stationary).

2. Palm Probability

Stochastic modelling involving prediction of the number of events (like births, accidental shocks, etc.) during a fixed period time $(0, t]$ or prediction of the waiting time distribution between two consecutive events or between any two events, say ith and jth event can be solved by an entirely new technique known as "Palm Probability". It is defined as the conditional probability of a specified number of events in a time interval given that an event has happened in the beginning of the interval.

Let $\phi_k(t)$ be the conditional probability of k number of shocks in $(0, t)$ given that a shock has occurred in the beginning of the interval ($\phi_k(t)$ is a Palm probability measure) and the $V_k(t)$ be the unconditional probability of k shocks in $(0, t]$, $k = 0, 1, 2, \ldots, n$. Using the Palm's integral equation which gives the relationship between $(\phi_k(t)$ and $V_k(t)$ as

$$V_k(t) = h(t) \int_0^t [\phi_{k-1}(\tau) - \phi_k(\tau)]\, d\tau$$

(where $h(t)$ = Intensity), the waiting time distribution of $(n+1)^{st}$ fatal shock given the time of $(r+1)^{st}$ non-fatal shock can be obtained for both time independent as well as time dependent intensities.

3. Develpoment of the Probability Model of the Waiting Time Distribution of the $(n+1)$th Fatal Shock given the Arrival of First $(r+1)$ Shocks at times $t_0 < t_1 < \ldots < t_r$ $(r = 0, 1, 2, \ldots, n-1)$

Denoting the random time of the $(n+1)$th shock which is fatal as T_n (a random variable), and assuming Markovity in the sequence of ordered shocks, we have

$$P(T_n \mid t_0, t_1, \ldots, t_r) = P(T_n \mid t_r), r = 0, 1, \ldots, n-1.$$

Given that at $T = t_r$, a shock arrives say, $(r+1)$th shock $(r = 0, 1, \ldots, n-1)$ with intensity λ and assuming that λ $(0 \leqslant \lambda < \infty)$ varies from individual to individual following the distribution as

$$\phi(\lambda) = \frac{a^k}{\Gamma(k)} e^{-a\lambda} \lambda^{k-1}, \quad 0 \leqslant \lambda < \infty$$
$$a, k > 0$$

the Palm probability distribution of the waiting time of $(n+1)$th shock given that the $(r+1)$th shock occurs at t_r, is given by (Biswas and Pachal (1983))

$$f_n(t_n \mid t_r) = \frac{(k+1)(k+2)\ldots(k+n-r)\, a^{k+1} (t_n - t_r)^{n-r-1}}{(n-r-1)!\, (a + t_n - t_r)^{n-r+k+1}} \tag{2}$$

$$\Rightarrow E_n(T_n \mid t_r) = E_n(T_{n-r} \mid t_0) = t_0 + \frac{(n-r)a}{k} \tag{3}$$

$$\text{Var}(T_n \mid t_r) = \frac{a^2(n-r)(n-r+k)}{k^2(k-1)}. \tag{4}$$

Next, we assume that the intensity is time dependent and given by $\lambda e^{-\delta t}$, $0 \leqslant t < \infty$. The intensity is increasing or decreasing according as $\delta < 0$ or $\delta > 0$ respectively.

Further, assuming that λ varies from individual to individual following

(1) and denoting the intensity at $T = t$ as

$$h(t \mid \lambda) = \lambda e^{-\delta t} \tag{5}$$

$$\Rightarrow h(t) = \int_0^\infty h(t \mid \lambda)\, \phi(\lambda) d\lambda$$

$$= \frac{k}{a}\, e^{-\delta t}. \tag{6}$$

Denoting by $V_0(t \mid \lambda)$ the conditional probability of no shock upto time $T = t$ given λ, we have

$$V_0(t \mid \lambda) = e^{-\int_0^t h(\tau \mid \lambda) d\tau} = e^{-\lambda A(t)} \tag{7}$$

where,

$$A(t) = \frac{1}{\delta}\,(1 - e^{-\delta t}). \tag{8}$$

Also $V_r(t \mid \lambda)$, the conditional probability of r shocks upto $T = t$ given λ is given by

$$V_r(t \mid \lambda) = \frac{(\lambda\, A(t))^r}{r\,!}\, e^{-\lambda A(t)} \tag{8'}$$

Hence,

$$V_0(t) = \int_0^\infty V_0(t \mid \lambda)\, \phi(\lambda)\, d\lambda$$

$$= \frac{a^k}{(a + A(t))^k}\,. \tag{9}$$

It may be seen that

$$\lim_{t -> 0} \frac{1 - V_0(t)}{t} = \frac{k}{a} \tag{10}$$

which shows the stationarity of the process. Also

$$\lim_{t -> 0} A(t) = 0 \tag{11}$$

and

$$\lim_{t -> 0} A'(t) = 1. \tag{12}$$

Define $G(z, t)$ and $G_0(z, t)$ as the probability generating functions (p.g.f) of $V_r(t)$ and $\phi_r(t)$ respectively.

Then, we have, from (8′)

$$G(z, t) = \sum_{r=0}^{\infty} z^r V_r(t) = \sum_{r=0}^{\infty} z^r \int_0^{\infty} V_r(t \mid \lambda)\, \phi(\lambda)\, d\lambda$$

$$= \sum_{r=0}^{\infty} \frac{a^k}{\Gamma(k)} \frac{(A(t))^r}{r\,!} \frac{\Gamma(r+k)}{(a + A(t))^{r+k}}$$

$$= \left[\frac{a}{a + (1 - z)\, A(t)}\right]^k, \quad \text{on simplification.} \tag{13}$$

Also using,

$$\frac{\partial}{\partial t} G(x, t) = -\frac{k}{a} (1 - z)\, G_0(z, t) \tag{14}$$

(vide Cox, and Isham, [4]).

$$\Rightarrow \left[\frac{a}{a + (1 - z)\, A(t)}\right]^{k+1} A'(t) = G_0(z, t). \tag{15}$$

Denoting by,

$$\overleftarrow{F_n(t)} = \sum_{N=0}^{\infty} \phi_N(t) \tag{16}$$

and

$$H_0(z, t) = \sum_{n=0}^{\infty} z^n \overleftarrow{F_n(t)} \tag{17}$$

$$\Rightarrow \frac{\partial}{\partial t} H_0(z, t) = \sum_{n=0}^{\infty} z^n \overleftarrow{f_n(t)}^{*}, \tag{18}$$

we have,

$$H_0(z, t) = \frac{1 - z\, G_0(z, t)}{1 - z}. \tag{19}$$

*$\overleftarrow{f_n(t)}$ represents the Palm probability density function of the nth shock further, given a shock occurs at time $T = 0$ and is in fact the waiting time distribution of $(n+1)^{st}$ shock since $T = 0$; $\overleftarrow{F_n(t)} = \int_0^t f_n(\tau)\, d\tau$ and $\overleftarrow{R_n(t)} = 1 - \overleftarrow{F_n(t)}$.

On substituting (15) in (19) and differentiating w.r.t. t, we have

$$\frac{\partial}{\partial t} H_0(z, t) = -\frac{z}{1-z} \frac{\partial}{\partial t}\left(\left(\frac{a}{a + A(t)(1-z)}\right)^{k+1} A'(t)\right)$$

$$= -\frac{z}{1-z} A''(t) \left(\frac{a}{a + A(t)(1-z)}\right)^{k+1} + \frac{z(k+1)\, a^{k+1} (A'(t))^2}{(a + A(t)(1-z))^{k+2}} \qquad (20)$$

$$= -z(1-z)^{-1} A''(t) \frac{a^{k+1}}{(a + A(t))^{k+1}}\Big[1 + (k+1) \frac{zA(t)}{a + A(t)} + \frac{(k+1)(k+2)}{1.2}\left(\frac{zA(t)}{a+A(t)}\right)^2 + \ldots + \frac{(k+1)\ldots(k+n-1)}{1.2\ldots(n-1)}\left(\frac{zA(t)}{a+A(t)}\right)^{n-1} + \frac{(k+1)\ldots(k+n)}{1.2\ldots n}\left(\frac{zA(t)}{z+A(t)}\right)^n + \ldots\Big]$$
$$+ \frac{z(k+1)\, a^{k+2} (A'(t))^2}{(a+A(t))^{k+2}}\Big[(1 + (k+2) \frac{zA(t)}{a+A(t)} + \frac{(k+2)(k+3)}{1.2}\left(\frac{zA(t)}{a+A(t)}\right)^2 + \ldots + \frac{(k+2)(k+3)\ldots(k+n)}{1.2 \quad (n-1)}\left(\frac{zA(t)}{a+A(t)}\right)^{n-1} + \ldots\Big] \qquad (21)$$

$$= \sum_{n=0}^{\infty} z^n \overbrace{f_n(t)}, \quad \text{on simplication.} \qquad (22)$$

Equating coefficients of z^n on both sides of (21), we get

$$\overbrace{f_n(t)} = \frac{a^{k+1} e^{-\delta t}}{(a + A(t))^{k+1}}\Big[\delta + \frac{(k+1)\delta\, A(t)}{a + A(t)} + \frac{(k+1)(k+2)\delta}{1.2}\left(\frac{A(t)}{a+A(t)}\right)^2 + \ldots + \frac{(k+1)\ldots(k+n-1)\delta}{(n-1)\,!}\left(\frac{A(t)}{a+A(t)}\right)^{n-1} + \frac{(k+1)\ldots(k+n)\, e^{-\delta t}}{(n-1)\,!} \frac{(A(t))^{n-1}}{(a+A(t))^n}\Big]. \qquad (23)$$

Finally using

$$E_n(T_n \mid t_0) = t_0 + \int_0^\infty (1 - \overleftarrow{F_n(t)})\, dt \qquad (24)$$

and

$$E(T_n \mid t_0, \ldots, t_r) = E(T_n \mid t_r) = t_r + \int_0^\infty (1 - \overleftarrow{F_{n-r}(t)})\, dt,$$

$$r = 0, 1, 2, \ldots, n-1 \qquad 25)$$

where,

$$\overleftarrow{F_n(t)} = 1 - \left[\frac{a^{k+1} A'(t)}{(a + A(t))^{k+1}} + \frac{(k+1)\, a^{k+1} A'(t)\, A(t)}{(a + A(t))^{k+2}} + \ldots + \frac{(k+1) \ldots (k+n-1)\, a^{k+1} A'(t) (A(t))^{n-1}}{(n-1)!\, (a + A(t))^{k+n}} \right]$$

$$= 1 - \overleftarrow{R_n(t)}$$

where $\overleftarrow{R_n(t)}$ is the survival function corresponding the c.d.f $F_n(t)$, we have

$$E_n(T_n \mid t_0) = \int_0^\infty \overleftarrow{R_n(t)}\, dt. \qquad (26)$$

Putting $n = 1, 2, 3, \ldots$ in (26) and straight forward integration makes

$$E_1(T_1 \mid t_0) = t_0 + \frac{a^{k+1}}{k} \left(\frac{1}{a^k} - \frac{1}{\left(a + \frac{1}{\delta}\right)^k} \right)$$

$$= f_1(a, k, \delta), \quad \text{say} \qquad (27)$$

$$E_2(T_2 \mid t_0) = t_0 + \frac{2a^{k+1}}{k} \left[\frac{1}{a^k} - \frac{1}{\left(a + \frac{1}{\delta}\right)^k} \right] - \frac{a^{k+1}}{\left(a + \frac{1}{\delta}\right)^k} + \frac{a^{k+2}}{\left(a + \frac{1}{\delta}\right)^{k+1}}$$

$$= f_2(a, k, \delta), \quad \text{say} \qquad (28)$$

$$E_3(T_3 \mid t_0) = t_0 + \frac{3a^{k+1}}{k} \left(\frac{1}{a^k} - \frac{1}{\left(a + \frac{1}{\delta}\right)^k} \right) - \frac{(k+5)\, a^{k+1}}{2\left(a + \frac{1}{\delta}\right)^k} + \frac{(k+3)\, a^{k+2}}{\left(a + \frac{1}{\delta}\right)^{k+1}} - \frac{(k+1)\, a^{k+3}}{2\left(a + \frac{1}{\delta}\right)^{k+2}}$$

$$= f_3(a, k, \delta), \quad \text{say} \qquad (29)$$

$$E_n(T_n \mid t_0) = t_0 + \frac{na^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{\left(a + \frac{1}{\delta}\right)^k}\right) + 0(\delta)$$

$$= f_n(a, k, \delta), \quad \text{say} \tag{30}$$

$$E_n(T_n \mid t_r) = E_n(T_{n-r} \mid t_0); \quad r = 0, 1, \ldots, n-1$$

$$= t_0 + \frac{(n-r)\, a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{\left(a + \frac{1}{\delta}\right)^k}\right) + 0\,(\delta) \tag{31}$$

when the number of shocks leading to failure is a random variable following geometric distribution with probability function

$$P(N = n) = q^{n-1}p, \quad n = 1, 2, 3, \ldots$$

where p is the probability of failure due to a particular shock then (3) and (31) can be rewritten as

$$E_n(T_n \mid t_r) = E_n(T_{n-r} \mid t_0)$$

$$= \sum_{n=1}^{\infty} E_{n-r}(T_{n-r} \mid t_0)\, P(N = n)$$

$$= \sum_{n=1}^{\infty}\left(t_0 + \frac{(n-r)\, a}{k}\right) q^{n-1}p$$

$$= t_0 + \frac{a}{k}\left(\frac{1}{p} - r\right) \tag{3'}$$

and

$$E(T_n \mid t_r) = E(T_{n-r} \mid t_0) = \sum_{n=1}^{\infty} E_{n-r}(T_{n-r} \mid t_0)\, P(N = n)$$

$$= \sum_{n=1}^{\infty}\left[t_0 + \frac{(n-r)\, a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{\left(a + \frac{1}{\delta}\right)^k}\right) + 0\,(\delta)\right] q^{n-1}p$$

$$= t_0 + \frac{a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{\left(a + \frac{1}{\delta}\right)^k}\right)\left(\frac{1}{p} - r\right) + 0(\delta). \tag{31'}$$

4. Estimation of the Parameters

To obtain estimates of a, k and δ from the observed data, the following technique may be used :

Initially assuming $\delta = 0$, the estimating equations, by the method of moments, are given by (3) and (4). While substituting the estimates of a and k so obtained in equation (27), a provisional estimate of δ can be obtained. Denoting all the estimates of a, k and δ so obtained by a_0, k_0 and δ_0, we can improve the estimates by successive iterations in the following way :

$$
\begin{aligned}
f_i(a, k, \delta) = f_i(a_0, k_0, \delta_0) &+ (a - a_0) \\
& \left.\frac{\partial f_i}{\partial a}\right|_{a_0, k_0, \delta_0} + (k - k_0) \left.\frac{\partial f_i}{\partial k}\right|_{a_0, k_0, \delta_0} \\
& + (\delta - \delta_0) \left.\frac{\partial f_i}{\partial \delta}\right|_{a_0, k_0, \delta_0} + 0(\,|\,a - a_0\,|\,) \\
& + 0(\,|\,k - k_0\,|\,) + 0(\,|\,\delta - \delta_0\,|\,), \\
& \qquad\qquad i = 1, 2, 3, \ldots \qquad (32)
\end{aligned}
$$

obtainable from equations (27), (28) and (29), which will give us three linear equations in a, k, δ starting from a_0, k_0, δ_0. Denoting the solutions of a, k, δ in the equations as a_1, k_1 and δ_1 respectively which are improved estimates of a_0, k_0, δ_0, the iteration procedure is carried on till we get reasonably good estimates of a, k and δ satisfying (27), (28) and (29).

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